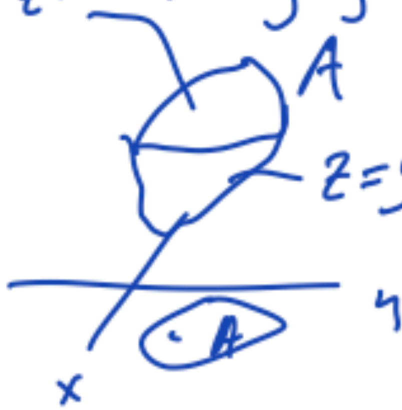
To compute volume between two surfaces:

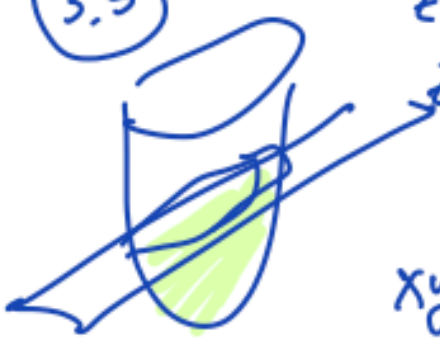
$z=f(x,y)$ $\iint (\text{top } z)-(\text{bottom } z)\, dA = \text{Volume}$

$\qquad\qquad\quad f(x,y) \qquad\quad g(x,y)$



$$= \iint_A \left(\int_{z=g(x,y)}^{z=f(x,y)} 1 \cdot dz \right) dA$$

3.5 Volume between



$z = 2x+9$ plane

$z = y^2+4y+x^2$ paraboloid point up

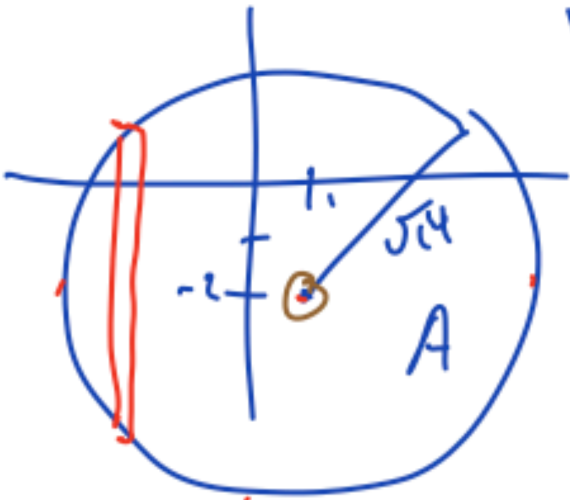
xy plane projection – region A of integration:

intersection: $z = 2x+9 = y^2+4y+x^2$

$$\underset{+5}{9} = \underset{+1}{x^2-2x} \quad \underset{+4}{y^2+4y}$$

$$14 = (x-1)^2 + (y+2)^2$$

circle of radius $\sqrt{14}$ centered at $(1,-2)$



Volume

$$= \iint_A \begin{pmatrix}\text{upper}\\ z\end{pmatrix} - \begin{pmatrix}\text{lower}\\ z\end{pmatrix}\, dA$$

$$= \iint_A (2x+9) - (y^2+4y+x^2)\, dA$$

$$(y+2)^2 = 14 - (x-1)^2$$

$$y+2 = \pm\sqrt{14-(x-1)^2}$$

$$y = -2 \pm \sqrt{14-(x-1)^2}$$

$x = 1-\sqrt{14}$ to $1+\sqrt{14}$

$$\int_{1-\sqrt{14}}^{1+\sqrt{14}} \int_{-2-\sqrt{}}^{-2+\sqrt{}} 2x+9-(y^2+4y+x^2)\, dy\, dx$$

polar version

$$x = 1 + r\cos\theta \qquad 0 \le r \le \sqrt{14}$$

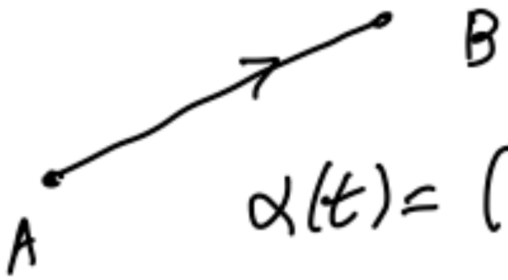
$$y = -2 + r\sin\theta \qquad 0 \le \theta \le 2\pi$$

$$dx\,dy = r\,dr\,d\theta$$

Integrating over Curves

Review: Parametrizing curves. (I'll use t for time as the parameter)

- Straight line between two points A & B.



$$\alpha(t) = (x(t), y(t), \ldots)$$

$$\alpha(t) = A + t(B - A)$$

example: $(1,3,-1)$ to $(2,7,0)$

$$\alpha(t) = (1,3,-1) + t\left[(2,7,0) - (1,3,-1)\right]$$

$$= (1,3,-1) + (t, 4t, t) \qquad (1,4,1)$$

$$\boxed{\alpha(t) = (\underbrace{1+t}_{x(t)}, \underbrace{3+4t}_{y(t)}, \underbrace{-1+t}_{z(t)}).} \quad 0 \le t \le 1$$

- We can always change the speed of a curve: replace t with some other function of t.

eg $\beta(t) = \alpha(t^3) \qquad 0 \leq t \leq 1$

$\beta(t) = (1+t^3, 3+4t^3, -1+t^3)$

$\gamma(t) = \alpha(100t)$ fast!

$\gamma(t) = (1+100t, 3+400t, -1+100t)$

$0 \leq t \leq .01$

- Any curve in $\mathbb{R}^2$ (eg function) let $t =$ one of the variables solve for the other.

→ eg. $y = x^5 - 2x^2 + 3$

$-1 \leq x \leq 6.$

let $x=t$ $\quad \alpha(t) = (t, t^5 - 2t^2 + 3)$

$-1 \leq t \leq 6$

→ eg. $y^3 - 3y = 4x^2 - 2$

let $y=t$ $\quad t^3 - 3t = 4x^2 - 2$

$$\frac{t^3 - 3t + 2}{4} = x^2$$

$$x = \pm \frac{\sqrt{t^3 - 3t + 2}}{2}$$

two pieces.

$$\alpha(t) = \left(\frac{\sqrt{t^3-3t+2}}{2}, t \right) \quad +x \text{ part}$$

$$\beta(t) = \left(-\frac{\sqrt{t^3-3t+2}}{2}, t \right) \quad -x \text{ part}$$

- circle $(x-1)^2 + (y+2)^2 = 14$

$$x = 1 + \sqrt{14} \cos \cancel{\theta}\, t$$
$$y = -2 + \sqrt{14} \sin \cancel{\theta}\, t$$

center $(1,-2)$
radius $\sqrt{14}$

$$0 \le t \le 2\pi$$

$$\alpha(t) = \left(1 + \sqrt{14}\cos(t), -2 + \sqrt{14}\sin t\right)$$

$0 \le t \le 2\pi$

(or $-\pi \le t \le \pi$)

Recall $\alpha'(t)$ = velocity vector →
tangent to the curve & points in
the direction where t increases.

Integrals over Curves

Two types:

(1) Integral of a function over the curve

$\int_\alpha (\text{density})\, ds$ = mass of a wire.

(2) Vector Integral of a Vector field over curve.

"Line Integral"

$\int_\alpha \vec{V} \cdot \vec{ds}$ = Work along α

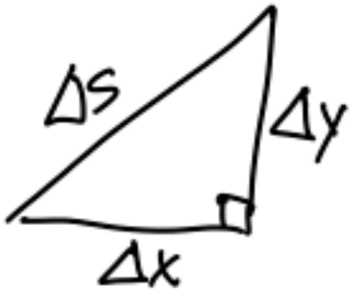
$\vec{V}$ = Force

(1) Integral of a function over a curve

$a \le t \le b$

$f(x, y, \ldots)$ given real valued function

$$\int_\alpha f\, ds = \lim_{\Delta s \to 0} \sum_i f(\alpha(t_i))\, \Delta s$$



$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2 + \dots}$$

Pythagorean theorem.

limit: $$ds = \sqrt{(dx)^2 + (dy)^2 + \dots}$$

To compute: $\alpha(t)$ curve

$a \le t \le b$

$$\alpha(t) = (x(t), y(t), \dots)$$

$$dx = x'(t)\,dt$$

$$dy = y'(t)\,dt$$

$$ds = \sqrt{dx^2 + dy^2}$$

$$= \sqrt{(x'(t))^2 dt^2 + (y'(t))^2 dt^2}$$

$$= \underbrace{\sqrt{(x'(t))^2 + (y'(t))^2 + \dots}}_{\|\alpha'(t)\|}\, dt$$

$$\int_\alpha f\,ds = \int_a^b f(\alpha(t))\, \underbrace{\|\alpha'(t)\|\,dt}_{ds}$$